When Eq. (23) is now substituted into Eq. (27) the result is

$$(XM^{-1}Y)_{ij} = \lim_{\lambda\to\infty} \frac{\lambda P_{ij}(\lambda)}{\delta(\lambda)} \tag{28}$$

The right-hand side of this last equation vanishes since $P_{ij}(\lambda)$ is of order $2n-2$, and $\delta(\lambda)$ is of order $2n$ in λ. Hence the ijth element of $XM^{-1}Y$ vanishes (regardless of the values of i and j). Consequently the product $XM^{-1}Y$ is a null matrix, and Eq. (12) is proved.

It is to be noticed that the proof just given does not depend on any restrictions imposed on the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$. The only requirement is that the system must have its $2n$ eigenvalues all distinct. Equation (12) is therefore applicable to nonconservative as well as conservative systems which meet this requirement. A numerical example is presented to illustrate these ideas.

Consider the free vibration of a two-degree-of-freedom nonconservative system whose equation is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} -4 & 1 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} 2 & -6 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{29}$$

Here the stiffness matrix is not symmetric, and the damping matrix is neither symmetric nor positve definite. On substituting a solution of the form $xe^{\lambda t}$, Eq. (29) yields the eigenvalue problem described by Eq. (1) where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} -4 & 1 \\ 3 & 3 \end{bmatrix}, \boldsymbol{C} = \begin{bmatrix} 2 & -6 \\ -1 & 6 \end{bmatrix} \tag{30}$$

The eigenvalues of the system are the roots of the characteristic equation

$$\det \boldsymbol{Q}(\lambda) \equiv \begin{vmatrix} \lambda^2 - 4\lambda + 2 & \lambda - 6 \\ 3\lambda - 1 & \lambda^2 + 3\lambda + 6 \end{vmatrix} = 0 \tag{31}$$

and on expanding the determinant, they are found to be

$$\lambda_1 = -2, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 3 \tag{32}$$

Corresponding to these eigenvalues, Eq. (31) shows that

$$\text{adj}\boldsymbol{Q}(\lambda_1) = \begin{bmatrix} 4 & 8 \\ 7 & 14 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}[1 \quad 2] \tag{33a}$$

$$\text{adj}\boldsymbol{Q}(\lambda_2) = \begin{bmatrix} 4 & 7 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}[4 \quad 7] \tag{33b}$$

$$\text{adj}\boldsymbol{Q}(\lambda_3) = \begin{bmatrix} 10 & 5 \\ -2 & -1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}[2 \quad 1] \tag{33c}$$

$$\text{adj}\boldsymbol{Q}(\lambda_4) = \begin{bmatrix} 24 & 3 \\ -8 & -1 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}[8 \quad 1] \tag{33d}$$

The eigenvectors and eigenrows can now be taken as any multiples of the columns and rows, respectively, of the right-hand side of Eqs. (33). If they are normalized in such a way that the first element of each is always unity, we have

$$x_1 = \begin{bmatrix} 1 \\ 7/4 \end{bmatrix}; x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}; x_3 = \begin{bmatrix} 1 \\ -1/5 \end{bmatrix}; x_4 = \begin{bmatrix} 1 \\ -1/3 \end{bmatrix} \tag{34a}$$

and

$$y_1^T = [1 \quad 2] \tag{34b}$$

$$y_2^T = [1 \quad 7/4] \tag{34c}$$

$$y_3^T = [1 \quad 1/2] \tag{34d}$$

$$y_4^T = [1 \quad 1/8] \tag{34e}$$

The normalization matrix in such a case is

$$\begin{aligned} \boldsymbol{M} &= \text{diag}\,\{y_i^T(2\lambda_i \boldsymbol{A} + \boldsymbol{B})x_i\}\ (i = 1,2,3,4) \\ &= \text{diag}\,\{-15/4, 2, -6/5, 5/3\} \end{aligned} \tag{35}$$

Equations (34) and (35) readily show that

$$XM^{-1}Y = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 7/4 & 1 & -1/5 & -1/3 \end{bmatrix}\begin{bmatrix} -4/15 & & & \\ & 1/2 & & \\ & & -5/6 & \\ & & & 3/5 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 7/4 \\ 1 & 1/2 \\ 1 & 1/8 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{36}$$

Equation (36) verifies the result predicted by Eq. (12).

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Integral Equation Methods in Duct Acoustics for Nonuniform Ducts with Variable Impedance

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Introduction

To handle nonuniform geometries and/or nonuniform wall linings one can go immediately to a numerical solution, as in the finite-difference method,[1-3] or one can

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solve the problem using any one of several analytic techniques (e.g., mode expansions,[4,5] weighted residuals,[6] perturbation methods[7]). An analytic method which has been overlooked in duct acoustics until recently is that of integral equations. Integral equation methods have been employed in electrostatics, hydrodynamics, diffraction, elasticity, and acoustic radiation.[8] It now appears that two approaches have emerged which use integral equations in duct acoustics. The work summarized here emphasized axially symmetric three-dimensional ducts with varying cross sections as indicated in Fig. 1. Moreover, variable wall linings which optimize the sound attenuation within both constant cross-section and variable cross-section ducts have been investigated.

A second approach which uses integral equation methods is that of Bell, Meyer, and Zinn.[11] Their approach differs from this author's in that they consider star shaped and rectangular cross sections in addition to the circular cross sections considered in this paper. However, the cross sections which they consider are constant functions of the axial direction while the cross sections in this paper vary as a function of the axial direction.

Equations

Although the integral equation method can be applied to partial differential equations of acoustics which contain mean-flow terms, the boundary conditions are complicated and care must be taken in dealing with them. Moreover, the mean-flow equations of Ref. 2 are not valid for variable cross-sectional ducts. Therefore, only the no-flow equation will be considered.

Of primary importance in potential theory is knowledge of a fundamental solution of the partial differential equation to be solved. (See the Appendix for a derivation of the fundamental solution of the acoustic equation for mean flow in rectangular duct). The fundamental solution for the three-dimensional axially symmetric duct involves an integral of the fundamental solution for the two-dimensional rectangular duct, which introduces an additional numerical complication. Hence, to simplify the numerics, only the two-dimensional, no-flow equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + (2\pi\eta)^2 p = 0 \tag{1}$$

will be considered, where x is the axial coordinate, y is the transverse coordinate, as shown in Fig. 1, and $\eta = Hf/c$ is the dimensionless frequency,[2] H is the height of the duct at the entrance, f is the frequency of the sound source, c is the speed of sound.

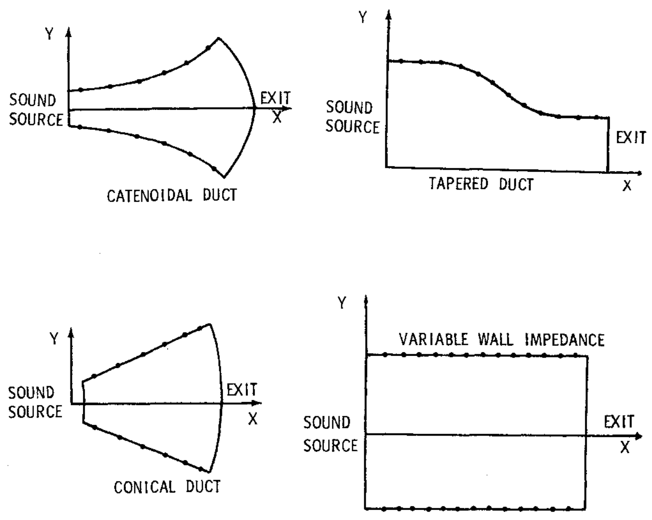

Fig. 1 Examples of variable area ducts.

The boundary conditions which must be satisfied in the no-flow case are as follows.

$$p(0,y) = f(y) \qquad \text{at the entrance} \tag{2a}$$

$$\frac{\partial p}{\partial n} = 0 \qquad \text{at } y=0 \text{ (the centerline)} \tag{2b}$$

$$\frac{\partial p}{\partial n} + \frac{i2\pi\eta p}{\zeta_w} = 0 \qquad \text{at the wall} \tag{2c}$$

$$\frac{\partial p}{\partial n} + \frac{i2\pi\eta p}{\zeta_e} = 0 \qquad \text{at the exit} \tag{2d}$$

where $(\partial p/\partial n)$ is the directional derivative in the direction of the outer normal vector $\boldsymbol{n}$ and equals $\boldsymbol{n} \cdot \nabla p$ by definition, and ζ_w and ζ_e are, respectively, the normalized wall and exit impedances; ($\zeta = Z/(\rho c)$ where Z is the wall impedance).

Method

Suppose that $k(x,y;s,t)$ is a fundamental solution of Eq. (1). This means that as a function of x and y, k is a solution of Eq. (1), where, for the purpose of this paper, the integration variable (s,t) is a point on the boundary of the duct separate from (x,y), which may be an interior or boundary point of the duct. In addition, for the points (x,y) and (s,t) sufficiently close together, k is required to approximately equal $k_0(x,y;s,t)$, the fundamental solution of Eq. (1) with $\eta = 0$, where

$$k_0(x,y;s,t) = -\log(r)/\pi$$

and

$$r = [(x-s)^2 + (y-t)^2]^{1/2}$$

Define the function $u(x,y)$ by

$$u(x,y) = \int_\Gamma k(x,y;s,t)\mu(s,t)\,\mathrm{d}S \tag{3}$$

where Γ is the boundary of the duct and μ is an unknown density function to be determined. Since the function k satisfies the differential equation as a function of x and y, one can differentiate under the integral sign at interior points of the duct to see that u satisfies the differential equation. Now this holds for arbitrary continuous densities μ. Hence, if μ is determined so that the boundary conditions are satisfied, then u is a solution of Eqs. (1) and (2). The particular fundamental solution to be employed is[12]

$$k(x,y;s,t) = -\tfrac{1}{2}Y_0(\alpha r) + i\tfrac{1}{2}J_0(\alpha r) \tag{4}$$

where $\alpha = 2\pi\eta$ and J_0 and Y_0 are Bessel functions of the first and second kind and zeroth-order, respectively. Moreover, for the points (x,y) and (s,t) close together (i.e., (x,y) is closed to the boundary), k is approximately equal to k_0. This facilitates the derivation of the integral equations. It is well-known in potential theory[8,10] that for the function u defined by Eq. (3) to satisfy a boundary conditon such as $(\partial u/\partial n) = f$, the density μ must satisfy the integral equation

$$\mu(x,y) + \int_\Gamma \frac{\partial}{\partial n} k(x,y;s,t)\mu(s,t)\,\mathrm{d}S = F(x,y)$$

at all points (x,y) belonging to the boundary. Therefore, the boundary conditions, Eq. (2), require that the density μ satisfy the integral equations

$$\int_\Gamma k(0,y;s,t)\mu(s,t)\,\mathrm{d}S = f(y) \qquad \text{at the entrance} \tag{5a}$$

$$\mu(x,y) + \int_\Gamma \hat{k}(x,y;s,t)\mu(s,t)\,dS = 0 \quad \text{on the remainder of the boundary} \qquad (5b)$$

where $\hat{k}(x,t;s,t) = (\partial/\partial n)k(x,y;s,t) + i\,2\pi\eta k(x,y;s,t)/\zeta$, and ζ, the normalized impedance, is a function of (x,y), the location on the boundary. For (x,y) at the centerline $\zeta=\infty$, at the exit $\zeta=\zeta_e$, and at the outer wall $\zeta=\zeta_w$.

This integral equation method can handle nonuniform cross sections and an axially varying impedance as follows. For nonuniform ducts one specifies Γ and the normal to Γ in the computer routine which solves the algebraic equations corresponding to Eq. (5). The variable wall impedance can be handled even more easily; one simply permits ζ to be a function of axial position on the outer wall.

Numerical Treatment

A set of points (s_m,t_m) $(m=1,M)$ are chosen around the boundary and the integrals in Eq. (5) are replaced by a quadrature formula

$$\int_\Gamma k(x,y;s,t)\mu(s,t)\,dS = \sum_{m=1}^{M} W_m k(x,y;s_m,t_m)\mu(s_m,t_m)$$

where W_m is the weight for the particular quadrature chosen. Now instead of determining the density $\mu(s,t)$ one simply determines the numbers $\mu_m = \mu(s_m,t_m)$. To solve for these M numbers one requires that Eqs. (5) are satisfied for $(x,y)=(x_m,y_m)$ $(m=1,M)$. In this way there are M equations and M unknowns to determine.

To be explicit, Eqs. (5) become

$$\sum_{\ell=1}^{M} W_\ell k(0,y_m;s_\ell,t_\ell)\mu(s_\ell,t_\ell) = f(y_m) \quad \text{for } 1 \le m \le M_0 \qquad (6a)$$

$$\mu(x_m,y_m) + \sum_{\ell=1}^{M} W_\ell \hat{k}(x_m,y_m;s_\ell,t_\ell)\mu(s_\ell,t_\ell) = 0 \quad \text{for } M_0 \le m \le M \qquad (6b)$$

where M_0 is the number of points at the entrance.

If the duct has a uniform cross section, then $t_\ell = 1$ at the outer wall and s_ℓ varies from the x coordinate of the entrance to that of the exit. For a duct with a nonuniform cross section neither s_ℓ nor t_ℓ will be constant on the outer wall. In this manner nonuniform cross sections are dealt with.

Results

Examples of variable area ducts which can be easily analyzed using the integral equation approach are depicted in Fig. 1. The marks on the walls of these two-dimensional ducts indicate typical placement of the quadrature points and hence typical points at which the wall lining is defined for an axially varying impedance. A key concern in the design of acoustic liners for jet engine nacelles is the determination of the liner yielding the greatest attenuation of sound, whether the liner is uniform or multisectional (variable). Since the attenuation within the duct is a function of the wall lining, one can obtain the best wall lining by maximizing the attenuation as a func-

Table 1 Comparison of the integral equation and finite-difference methods uniform duct, $L/H = 0.5$, $\eta = 1$, $\zeta_w = 0.388 - 0.846i$

Method	Number of Mesh Points		Total	Attenuation (dB)	Time (sec)
	In x direction	In y direction			
Finite difference	10	10	100	−2.8	2.8
	20	20	400	−3.39	6.
Integral equation	10	10	40	−3.51	0.8[a]
	20	10	60	−3.48	1.5[b]
	20	20	80	−3.52	3.8[c]

[a]The setup time, which is required only once for a particular geometry even though 50 or more evaluations are required for an optimization, is an additional 2.4 sec. To compute the pressures at 100 points within the duct requires an additional 1.6 sec. Again, this is unnecessary when performing an optimization.

[b]The setup time is 3.0 sec and the time required to compute pressures at 200 points within the duct is 4.5 sec in this case.

[c]The setup time is 3.6 sec while the time required to compute pressures at 400 points within the duct is 14.5 sec in this case.

Table 2 Uniform and three sectional linings yielding largest sound attenuation for several different variable area ducts

Duct type	Uniform lining	Attenuation (dB)	Three sectional lining	Attenuation (dB)
Conical[a]	$0.247-0.770i$	−1.725	$\zeta_1 = 0.00003-0.897i$	
			$\zeta_2 = 0.0166-0.533i$	−4.5
			$\zeta_3 = 0.0095-0.146i$	
Sinusoidal[b]	$0.468-0.897i$	−9.898	$\zeta_1 = 0.00004-0.982i$	
			$\zeta_2 = 0.0750-1.44i$	−27.5
			$\zeta_3 = 0.208-0.178i$	
Catenoidal[c]	$0.213-0.736i$	−5.893	$\zeta_1 = 0.0001-0.858i$	
			$\zeta_2 = 0.413-1.057i$	−10.3
			$\zeta_3 = 0.226-0.341i$	

[a]The equation of the outer wall is $y=(x+2)/2$ for $0 \le x \le 1$, so height at the entrance is 1.

[b]The equation of the outer wall is

$$\begin{aligned} y &= 0.6 && \text{for } 0 \le x \le 0.2 \\ &= 0.5 + 0.1\cos(\pi x/0.6 - \pi/3) && \text{for } 0.2 < x < 0.8 \\ &= 0.4 && \text{for } 0.8 \le x \le 1. \end{aligned}$$

so that the height at the entrance is 0.6.

[c]The equation of the outer wall is $y = 0.25\,[\exp(1.25x) + \exp(-1.25x)]$, so that the height at the entrance is 0.5.

tion of wall lining. For the ducts shown, the impedance of the duct walls could assume ten or twelve different values and a standard minimization routine would compute the combination of ten or twelve impedances yielding the greatest attenuation of sound levels within the duct. In actual practice, to limit computation time, one requires the impedance to be constant on an entrance section of the duct (made up of, say, four integration points), then permits the impedance to change to a second value on a middle section (comprised of four or more integration points), and then have a third value on a termination section (consisting of another four integration points). Then the minimization routine minimizes a function of three complex or six real variables, a much less time-consuming effort than minimizing a function of 24 real variables.

To compare the computation times of the integral equation approach versus that of the finite-difference approach, a uniform duct with $L/H=0.5$ (L and H are the duct length and height), $\eta=1$, and $\zeta_w=0.388-0.846i$ was analyzed. The study included different sets of integration points around the boundary of the duct for the integral equation method and 100 and 400 mesh points for the finite-difference method. The results are summarized in Table 1.

The conclusion drawn from an inspection of Table 1 is that the integral equation method can give comparable accuracy with much less computational time than the finite-difference method. A key factor is that the finite-difference method computes the pressure distribution within the entire duct even though the acoustic intensities are required only at the entrance and the exit. However, in the integral equation method, an appreciable savings in computational time is obtained by computing only the desired quantities at the entrance and the exit.

For the optimizations to be carried out, the setup time occurred only once even if 50 different impedance values were considered. Hence, the time required is less than 1 sec per evaluation or 50 sec total for an optimization requiring 50 function evaluations. This compares with 300 sec of computation time for a comparable study using finite differences. All the computations were carried out on a CDC 6600 computer.

The results for three of the ducts depicted in Fig. 1 are summarized in Table 2. The conical duct was chosen because the exact solution is known for a hard walled duct. The tapered duct was chosen because it has not been handled by the conformal mapping/finite-difference method and is extremely easy to handle using integral equations. The catenoidal duct is more difficult. It has not been solved by the conformal mapping/finite-difference method either. The equation of the outside wall is straightforward because it is a catenary. However, the exit curve (or surface) is more difficult and is dealt with numerically.

For all three example cases considered, a three-sectional liner was found which yielded twice to three times the attenuation of the best uniform liner. This agrees with results obtained in Ref.2.

Appendix: Fundamental Solutions of the Partial Differential Equations Arising in Duct Acoustics

The fundamental solution for the partial differential equation of duct acoustics for a duct containing mean flow, that is, for an equation of the form

$$(1-M^2)\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}-4\pi\eta Mi\frac{\partial p}{\partial x}+(2\pi\eta)^2 p=0 \qquad \text{(A1)}$$

is

$$p(x,y)=e^{(ax)}H_0^1(wr) \qquad \text{(A2)}$$

where

$$a=i2\pi\eta M/(1-M^2)$$

$$w=(2\pi\eta)^2+a^2(1-M^2)-i4\pi\eta Ma$$

$$r=[(x/[1-m^2]^{1/2}-s)^2+(y-t)^2]^{1/2}$$

$$H_0^1=J_0+iY_0$$

(See Eq. (4) and Ref. 12). This is verified by direct substitution of Eq. (A2) into Eq. (A1).

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Derivation of an Integral Equation for Transonic Flows

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I. Introduction

THE nonlinear partial differential equation for the perturbation velocity potential and the boundary conditions

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